Polynomial two-parameter eigenvalue problems and matrix pencil methods for stability of delay-differential equations

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Abstract

Several recent methods used to analyze *asymptotic stability* of *delay-differential equations* (DDEs) involve determining the eigenvalues of a matrix, a matrix pencil or a *matrix polynomial* constructed by *Kronecker products*. Despite some similarities between the different types of these so-called *matrix pencil methods*, the general ideas used as well as the proofs differ considerably. Moreover, the available theory hardly reveals the relations between the different methods.

In this work, a different derivation of various matrix pencil methods is presented using a unifying framework of a new type of eigenvalue problem: the *polynomial two-parameter eigenvalue problem*, of which the *quadratic two-parameter eigenvalue problem* is a special case. This framework makes it possible to establish relations between various seemingly different methods and provides further insight in the theory of matrix pencil methods.

We also recognize a few new matrix pencil variants to determine DDE stability. Finally, the recognition of the new types of eigenvalue problem opens a door to efficient computation of DDE stability.

Key words: Delay-differential equations, two-parameter eigenvalue problem, multiparameter eigenvalue problem, critical delays, robustness, stability, asymptotic stability, companion form, quadratic eigenvalue problem, polynomial eigenvalue problem, quadratic two-parameter eigenvalue problem, polynomial two-parameter eigenvalue problem.

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1 Introduction

Mathematical models consisting of *delay-differential equations (DDEs)*, in the simplest form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau), \quad A_0, A_1 \in \mathbb{C}^{n\times n}, \quad \tau \geq 0, \tag{1}$$

occur naturally in a wide variety of fields related to applied mathematics, such as engineering, control theory, biology, traffic modeling, neural networks, mechanics, and electronic circuits. For most applications, it is desired that $x(t) \to 0$ as $t \to \infty$ for any bounded initial condition. This property is referred to as *asymptotic stability.* If a DDE is not asymptotically stable, it is typically not an accurate model of reality, or the modeled system has unwanted properties, such as oscillations, or the energy content of the system is unbounded in time causing the modeled physical object to break or at least turn inefficient. Clearly, asymptotic stability is important in practice, and numerical and analytical tools to analyze asymptotic stability of DDEs are popular topics of research. For instance, large parts of several monographs, which are standard references in the field of DDEs, deal with stability of DDEs, e.g., the books of Bellman [2], Niculescu [25], Michiels and Niculescu [22] and Gu, Kharitonov, and Chen [9]. See also the survey papers [27], [10], and [18]. This paper is concerned with asymptotic stability for (1) as well as more general DDEs: DDEs with delays in the derivative (called *neutral DDEs*) and DDEs with multiple delays. We mention that DDEs that are not of neutral type are also called *retarded.*

Asymptotic stability is often described using the solutions of the characteristic equation associated with (1):

$$\det(-\lambda I + A_0 + A_1 e^{-\tau\lambda}) = 0,$$

of which the solutions λ are called eigenvalues; the set of eigenvalues is called the *spectrum.* The DDE (1) is asymptotically stable if and only if the spectrum is contained in the open left half plane (see, e.g., [22, Prop. 1.6]).

We will also consider more general classes of DDEs in this paper. For some of these DDEs, in particular neutral DDEs, it is not sufficient that all eigenvalues have negative real parts to ensure asymptotic stability. A neutral DDE is asymptotically stable if and only if the supremum of the real part of the spectrum is negative (see, e.g., [22, Prop. 1.20]).

Because of these relations between asymptotic stability and the spectrum, explicit conditions such that there is a purely imaginary eigenvalue can be very useful in a stability analysis. In this paper we will study explicit conditions on the delay τ such that there is at least one purely imaginary eigenvalue.

In the literature, there are several approaches to characterize these values of τ, sometimes called *critical delays*, *switching delays*, *crossing delays*, or *kernel and offspring curves*; see [14, Remark 3.1] for some comments on terminology.

One approach to determine critical delays is to consider the eigenvalues of certain matrices or matrix pencils constructed by Kronecker products. Methods of this type are presented by Chen, Gu, and Nett [4] (see also [9, Thm. 2.13]), Louisell [19], Niculescu [24] (see also [22, Prop. 4.5]), and Fu, Niculescu, and Chen [8, 26]. The works [15] (see also [14, Chapter 3]), [13], and [6] also use a formulation of eigenvalue problems containing Kronecker products. Even though these popular methods have some characteristics in common, the ideas used in the derivations differ. For instance, Louisell [19] derives a result for neutral DDEs by considering a linear ODE which is proven to share imaginary eigenvalues with the DDE whereas Chen, Gu, and Nett [3] and several other authors depart from the characteristic equation and exploit the fact that the eigenvalues of Kronecker products are products of the eigenvalues of the individual factor matrices.

For given matrices $A_i, B_i, C_i \in \mathbb{C}^{n\times n}$, $i = 1, 2$, the (linear) two-parameter eigenvalue problem is concerned with finding $\lambda, \mu \in \mathbb{C}$ and $x, y \in \mathbb{C}^n\backslash\{0\}$ such that

$$\begin{cases} A_1x = \lambda B_1x + \mu C_1x, \\ A_2y = \lambda B_2y + \mu C_2y. \end{cases} \tag{2}$$

There is a close connection between linear two-parameter problems and two coupled generalized eigenvalue problems involving Kronecker products; see [1] and Section 3 for further details.

In this paper, we will consider *polynomial two-parameter eigenvalue problems* and show that there are associated (one-parameter) quadratic eigenvalue problems which are very relevant for critical delays of DDEs. Note that the use of *multivariate polynomials*, which are closely related to multiparameter eigenvalue problems, is not new in the field of stability of DDEs. Multivariate polynomials are used in, e.g., [17], [16], and [11] with applications in [5]; see also the summaries in the standard references [25, Section 4.1.2] and [9, Section 4.6]. In this work, we discuss a new natural way to interpret matrix pencil methods in the context of two-parameter eigenvalue problems.

The goals of this paper are:

(a) to introduce a new type of eigenvalue problem, the *polynomial two-parameter eigenvalue problem*, with the *quadratic two-parameter eigenvalue problem* as important special case;
(b) to show the relevance of this problem to determine critical delays for various

types of DDEs;

(c) to provide alternative derivations of existing matrix pencil methods using the context of polynomial two-parameter eigenvalue problems;

(d) to hereby provide a new unifying framework for the determination of critical delays;

(e) and, finally, to recognize a few new variants of known matrix pencil methods.

The results of this work are ordered by increasing generality. The idea of an alternative interpretation of matrix pencil methods using polynomial two-parameter eigenvalue problems is first illustrated in Section 2. In Section 3 we give connections between certain quadratic two-parameter eigenvalue problems and associated quadratic (one-parameter) eigenvalue problems. These links are used to derive the polynomial eigenvalue problem occurring in matrix pencil methods for more general types of DDEs in Section 4. After stating some new variants of matrix pencil methods in Section 5, we end with some conclusions and an outlook in Section 6.

2 DDEs with a single delay

An important aspect of this work is a further understanding of matrix pencil methods. The derivation for the most general type of DDE is somewhat technical and contains expressions difficult to interpret by inspection. Therefore, to ease the presentation, it is worthwhile to first illustrate the general ideas of the theory by considering retarded DDEs with a single delay.

In this section we derive a polynomial two-parameter eigenvalue problem corresponding to purely imaginary eigenvalues of a DDE, and apply a result that will be proved in Theorem 3 in Section 3 to identify that the eigenvalue problems are the ones that occur in the matrix pencil methods proposed in [4] and [19].

First, we introduce the following (usual) notations: $\sigma(A)$ and $\sigma(A, B)$ denote the spectrum of a matrix A and matrix pencil (A, B), respectively; I denotes the identity matrix, $\otimes$ the Kronecker product and $\oplus$ the Kronecker sum (i.e., $A \oplus B = A \otimes I + I \otimes B$). If a DDE is stable for $\tau = 0$, then τ_* denotes the *delay margin*, i.e., the smallest delay τ for which the DDE is no longer stable.

Consider the DDE

$$B_0\dot{x}(t) = A_0x(t) + A_1x(t - \tau), \tag{3}$$

where $A_0, A_1, B_0 \in \mathbb{C}^{n\times n}$. We will rederive the eigenvalue problems that arise in the following two matrix pencil results. We hereby note that matrix pencil methods are generally stated in various degrees of generality, for various types of DDEs. Theorem 1 below is for the slightly different setting of neutral DDEs and $B_0 = I$. Theorem 2 as stated here is a restriction of the original result [4] to single delays. We postpone the discussion of the more general result in [4] to Section 4.

Theorem 1 (Louisell [19, Theorem 3.1]) *Let $A_0, A_1, B_1 \in \mathbb{R}^{n\times n}$. Then all purely imaginary eigenvalues of the neutral DDE*

$$\dot{x}(t) + B_1\dot{x}(t-\tau) = A_0x(t) + A_1x(t-\tau) \tag{4}$$

are zeros of

$$\det((\lambda I - A_0) \otimes (\lambda I + A_0) - (\lambda B_1 - A_1) \otimes (\lambda B_1 + A_1)) = 0. \tag{5}$$

Theorem 2 (Chen, Gu, and Nett; special case of [4, Theorem 3.1]) *Suppose* (1) *is stable for $\tau = 0$. Define*

$$U := \begin{bmatrix} I & 0 \\ 0 & A_1 \otimes I \end{bmatrix} \quad \textit{and} \quad V := \begin{bmatrix} 0 & I \\ -I \otimes A_1^T & -A_0 \oplus A_0^T \end{bmatrix}. \tag{6}$$

If the delay margin τ_* is finite and nonzero, then $\tau_* = \min_k \frac{\alpha_k}{\omega_k}$ where $\alpha_k \in [0, 2\pi]$, $\omega_k > 0$, and $e^{-i\alpha_k} \in \sigma(V, U)$ satisfies the relation $i\omega_k \in \sigma(A_0 + A_1e^{-i\alpha_k})$.

Theorem 2 gives a formula for the delay margin in terms of the solutions of the generalized eigenvalue problem involving the pencil (V, U), which represents a linearization of the *quadratic eigenvalue problem* (QEP)

$$\left(\mu^2(A_1 \otimes I) + \mu\,(A_0 \oplus A_0^T) + I \otimes A_1^T\right) v = 0 \tag{7}$$

for $\mu \in \mathbb{C}$ and nonzero $v \in \mathbb{C}^{n^2}$. An exhaustive characterization of possible linearizations was recently given in [20]. Observe that the matrix polynomial (5) in Theorem 1 also represents a quadratic eigenvalue problem; a linearization was given in [19] as well. (A linearization adapted to the quadratic eigenvalue problem in the matrix pencil method in [15] was recently proposed in [7].)

Both of the matrix pencil methods (Theorem 1 and Theorem 2) involve quadratic eigenvalue problems (5) and (7). However, from the original derivation and proofs of these results, there is no obvious relation between these two approaches. We will develop a framework that can derive both quadratic eigenvalue problems in a unifying manner which gains further insight in the relations between the methods.

Consider the eigenvalue problem associated with (3)

$$\lambda B_0x = (A_0 + A_1e^{-\lambda\tau})\, x, \tag{8}$$

for nonzero $x \in \mathbb{C}^n$. We are interested in the case where there is a purely imaginary eigenvalue, say $\lambda = i\omega$. We denote $\mu = e^{-\lambda\tau}$. Under the assumption that the eigenvalue is imaginary, i.e., $\lambda = i\omega$, we have $\overline{\lambda} = -\lambda$ and $\overline{\mu} = \mu^{-1}$. This yields

$$-\lambda \overline{B}_0 y = (\overline{A}_0 + \mu^{-1}\overline{A}_1)\, y, \tag{9}$$

where $y = \overline{x}$. Hence, multiplying (9) by μ and rearranging the terms, we have

$$\begin{cases} A_0 x = \lambda B_0 x \quad -\mu A_1 x, \\ \overline{A}_1 y = -\lambda\mu \overline{B}_0 y \ -\mu \overline{A}_0 y. \end{cases} \tag{10}$$

Now first, for given A_i, B_i, C_i, D_i, E_i, and $F_i \in \mathbb{C}^{n\times n}$, $i = 1, 2$, consider the following *quadratic two-parameter eigenvalue problem*

$$\begin{cases} (A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)\, x = 0, \\ (A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)\, y = 0, \end{cases}$$

where the assignment is to compute one or more tuples (λ, μ, x, y) with nonzero x and y. As for the linear two-parameter eigenvalue problem we will call (λ, μ) an eigenvalue and $x \otimes y$ an eigenvector. We see that (10) is a special case of this general quadratic two-parameter eigenvalue problem with just one nonlinear term and one additional vanishing matrix. Theorem 3 in the next section implies that the following two (one-parameter) quadratic eigenvalue problems are associated with (10):

$$\left[\lambda^2 (B_0 \otimes \overline{B}_0) + \lambda (B_0 \otimes \overline{A}_0 - A_0 \otimes \overline{B}_0) + (A_1 \otimes \overline{A}_1 - A_0 \otimes \overline{A}_0)\right] (x \otimes y) = 0 \tag{11}$$

and

$$\left[\mu^2 (A_1 \otimes \overline{B}_0) + \mu\, (A_0 \otimes B_0 + B_0 \otimes \overline{A}_0) + (B_0 \otimes \overline{A}_1)\right] (x \otimes y) = 0. \tag{12}$$

Using these QEPs, we can now rederive Theorems 1 and 2 as follows. Although Theorem 1 applies to the wider class of neutral DDEs, we can restrict it to the class of retarded DDEs by setting $B_1 = 0$. Then taking $B_0 = I$ in the quadratic eigenvalue problem (11) exactly renders (5) in Theorem 1, under the assumption that the matrices are real.

Similarly, relating Theorem 2, (12) corresponds to the quadratic eigenvalue problem (7); note that (12) gives (7) if we replace conjugation by the conjugate transpose as follows. Instead of (9) as the conjugate of (8), we can also take the conjugate transpose of (8). (In Section 5, we will exploit similar techniques to derive new matrix pencil methods.) The resulting quadratic two-parameter eigenvalue problem is

$$\begin{cases} A_0 x = \lambda B_0 x & -\mu A_1 x, \\ A_1^* y = -\lambda\mu B_0^* y & -\mu A_0^* y, \end{cases} \tag{13}$$

where now y is the left eigenvector of (8). The second equation in (13) is the transpose of the second equation in (10). Application of Theorem 3 to (13) yields (7). In the next section we will prove Theorem 3 which implies (11) and (12) as well as more general results.

3 Quadratic two-parameter eigenproblems and associated quadratic one-parameter eigenproblems

First, we will review some facts for the (linear) two-parameter eigenvalue problem (2), see also [1]. Define the matrix determinants

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2, \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2, \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2, \end{aligned}$$

where $\Delta_i \in \mathbb{C}^{n^2 \times n^2}$, $i = 0, 1, 2$. Associated with (2) are two (decoupled) generalized eigenvalue problems (GEPs)

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z, \\ \Delta_2 z &= \mu \Delta_0 z, \end{aligned} \tag{14}$$

where $z = x \otimes y$. (In fact, these GEPs are equivalent with (2) if Δ_0 is nonsingular; see [1]). There are two generalized eigenvalue problems which correspond to the linear two-parameter eigenvalue problem (2); likewise, we will see that there are two quadratic (one-parameter) eigenvalue problems which correspond to the quadratic two-parameter eigenvalue problem. We will see in the derivations of the matrix pencil methods that some methods correspond to one form and some to the other.

To be able to handle a wider classes of DDEs, we will prove a result that deals with a generalization of problem (10); we will make use of this theorem in the derivation of matrix pencil methods in Section 4. The generalization of (10), which we will use for neutral DDEs, involves an additional cross term $\lambda\mu$:

$$\begin{cases} A_1x = \lambda B_1x + \mu C_1x + \lambda\mu D_1x, \\ A_2y = \lambda B_2y + \mu C_2y + \lambda\mu D_2y. \end{cases} \tag{15}$$

Theorem 3 *If (λ, μ) is a solution of (15) with corresponding eigenvector (x, y) then:*

1. *λ is an eigenvalue with corresponding eigenvector $x \otimes y$ of the QEP*

$$\begin{aligned}\Big[\lambda^2(D_1 \otimes B_2 - B_1 \otimes D_2) + \lambda(A_1 \otimes D_2 - D_1 \otimes A_2 \\ - B_1 \otimes C_2 + C_1 \otimes B_2) + (A_1 \otimes C_2 - C_1 \otimes A_2)\Big](x \otimes y) = 0.\end{aligned} \tag{16}$$

2. *μ is an eigenvalue with corresponding eigenvector $x \otimes y$ of the QEP*

$$\begin{aligned}\Big[\mu^2(D_1 \otimes C_2 - C_1 \otimes D_2) + \mu\,(A_1 \otimes D_2 - D_1 \otimes A_2 \\ - C_1 \otimes B_2 + B_1 \otimes C_2) + (A_1 \otimes B_2 - B_1 \otimes A_2)\Big](x \otimes y) = 0.\end{aligned} \tag{17}$$

Proof. We show the first implication; the second follows by switching the roles of λ and μ; B_1 and B_2; and C_1 and C_2.

Equation (16) holds because

$$\begin{aligned}&\lambda^2(D_1 \otimes B_2 - B_1 \otimes D_2)(x \otimes y) \\ &\quad = \lambda\,(D_1 \otimes (A_2 - \mu C_2 - \lambda\mu D_2) - (A_1 - \mu C_1 - \lambda\mu D_1) \otimes D_2)(x \otimes y) \\ &\quad = \lambda\,(D_1 \otimes (A_2 - \mu C_2) - (A_1 - \mu C_1) \otimes D_2)(x \otimes y) \\ &\quad = (\lambda\,(D_1 \otimes A_2 - A_1 \otimes D_2) + \lambda\mu\,(C_1 \otimes D_2 - D_1 \otimes C_2))(x \otimes y) \\ &\quad = (\lambda\,(D_1 \otimes A_2 - A_1 \otimes D_2) + (C_1 \otimes (A_2 - \lambda B_2) - (A_1 - \lambda B_1) \otimes C_2))(x \otimes y),\end{aligned}$$

where we used that

$$\begin{aligned}&\lambda\mu\,(C_1 \otimes D_2 - D_1 \otimes C_2)(x \otimes y) \\ &\quad = (C_1 \otimes (A_2 - \lambda B_2 - \mu C_2) - (A_1 - \lambda B_1 - \mu C_1) \otimes C_2)(x \otimes y) \\ &\quad = (C_1 \otimes (A_2 - \lambda B_2) - (A_1 - \lambda B_1) \otimes C_2)(x \otimes y).\end{aligned}$$

□

Theorem 3 not only explains (11) and (12) but can also be used to derive results for neutral DDEs in the next section.

4 Generalizations for neutral systems and multiple delays

In Section 2 we used the quadratic two-parameter eigenvalue problem (10) to derive the quadratic eigenvalue problems in Theorems 1 and 2. However, in Section 2 we limited ourselves to the setting of a single delay DDE. The original formulations of Theorem 1 [19] and Theorem 2 [4], were stated for more general types of DDEs, which we will study in this section. In particular, we discuss neutral systems in Section 4.1 and the DDEs with multiple commensurate DDEs in Section 4.2.

4.1 Neutral DDEs

Consider the neutral DDE

$$B_0\dot{x}(t) + B_1\dot{x}(t-\tau) = A_0x(t) + A_1x(t-\tau),$$

where $A_0, A_1, B_0, B_1 \in \mathbb{C}^{n\times n}$. The generality of Theorem 3 allows us to derive the matrix pencil methods for this DDE in a similar way as in Section 2. With $\lambda = i\omega$ and $\mu = e^{-i\tau\omega}$ we note that the corresponding eigenvalue problem and its complex conjugate can be expressed as

$$\begin{cases} A_0x = \lambda B_0x + \lambda\mu B_1x - \mu A_1x, \\ \overline{A}_1y = -\lambda\overline{B}_1y - \lambda\mu\overline{B}_0y - \mu\overline{A}_0y. \end{cases} \tag{18}$$

After applying Theorem 3 we derive that

$$\Big[\lambda^2(-B_1\otimes\overline{B}_1 + B_0\otimes\overline{B}_0) + \lambda(-A_0\otimes\overline{B}_0 - B_1\otimes\overline{A}_1 + B_0\otimes\overline{A}_0 + A_1\otimes\overline{B}_1) + (-A_0\otimes\overline{A}_0 + A_1\otimes\overline{A}_1)\Big](x\otimes y) = 0,$$

and after rearranging the terms we get

$$\Big((\lambda B_0 - A_0)\otimes(\lambda\overline{B}_0 + \overline{A}_0) - (\lambda B_1 - A_1)\otimes(\lambda\overline{B}_1 + \overline{A}_1)\Big)(x\otimes y) = 0. \tag{19}$$

This is a slight generalization of the eigenvalue problem presented by Louisell [19], since in [19] it is assumed that $B_0 = I$ and that the matrices are real. Louisell, motivated by a connection with a certain differential equation of which all purely imaginary eigenvalues coincide with purely imaginary eigenvalues of the DDE, suggests that (5) can be determined by solutions of the generalized eigenvalue problem

$$\lambda\begin{bmatrix} I\otimes I & B_1\otimes I \\ I\otimes B_1 & I\otimes I \end{bmatrix} w = \begin{bmatrix} A_0\otimes I & A_1\otimes I \\ -I\otimes A_1 & -I\otimes A_0 \end{bmatrix} w.$$

We note that this is just one possible linearization of (5); any of the linearizations in [20, 21] might be considered. Moreover, there also exist numerical methods for quadratic eigenvalue problems that try to avoid linearization; see [28] for an overview.

The second resulting quadratic eigenvalue problem from applying Theorem 3 to (18) reads

$$\Big[\mu^2(B_1 \otimes \overline{A}_0 + A_1 \otimes \overline{B}_0) + \mu\,(A_0 \otimes \overline{B}_0 + B_1 \otimes \overline{A}_1 + \\ + A_1 \otimes \overline{B}_1 + B_0 \otimes \overline{A}_0) + (A_0 \otimes \overline{B}_1 + B_0 \otimes \overline{A}_1)\Big]\,(x \otimes y) = 0.$$

At this point we note that we can interchange all left and right operators in the Kronecker products to get a special case of the result in [13] (where multiple delays are considered).

To determine a relation with a result by Fu, Niculescu, and Chen [8] we note that similarly to the derivation of (18) using x and $y = \overline{x}$, we can also derive a quadratic two-parameter eigenvalue problem involving x and its corresponding left eigenvector y:

$$\begin{cases} A_0 x = \lambda B_0 x + \lambda\mu B_1 x - \mu A_1 x, \\ A_1^* y = -\lambda B_1^* y - \lambda\mu B_0^* y - \mu A_0^* y. \end{cases} \tag{20}$$

Application of Theorem 3 yields

$$\Big[\mu^2(B_1 \otimes A_0^* + A_1 \otimes B_0^*) + \mu\,(A_0 \otimes B_0^* + B_1 \otimes A_1^* + \\ + A_1 \otimes B_1^* + B_0 \otimes A_0^*) + (A_0 \otimes B_1^* + B_0 \otimes A_1^*)\Big]\,(x \otimes y) = 0.$$

This is a special case of the method in [8] which applies to DDEs with multiple commensurate delays, which will be the topic of the next subsection [2].

4.2 Multiple delays

We now consider the case of DDEs with multiple, say $m > 1$, delays. In the literature there are essentially two ways to handle this situation. Either the curves or surfaces corresponding to the critical delays are parameterized using $m - 1$ free variables, as is done in for instance [6, 13, 15]. In other approaches, e.g., [4, 8], it assumed that the delays are *commensurate*: multiple delays that are all integer multiples of some delay value τ, i.e., $\tau_k = \tau n_k$ where $n_k \in \mathbb{N}$.

[2] Note that B_1 is defined with an opposite sign in [8].

Here, we will focus on the case of multiple commensurate delays as the parameterization approach does not yield stability information from the solution of just one eigenvalue problem.

Consider the DDE with commensurate delays

$$B_0\dot{x}(t) = \sum_{k=0}^{m} A_k x(t - \tau k).$$

The associated eigenvalue problem is

$$\left(\sum_{k=0}^{m} e^{-\tau k\lambda} A_k - \lambda B_0\right) v = 0.$$

As in the previous section we substitute $\lambda = i\omega$ and $\mu = e^{-i\tau\omega}$ and consider the complex conjugate of the eigenvalue problem. After rearrangement of the terms and sums we have

$$\begin{cases} -\overline{A}_m u = \lambda \mu^m \overline{B}_0 u + \sum_{k=1}^{m} \mu^k \overline{A}_{m-k} u, \\ A_0 v \ = \lambda B_0 v \qquad - \sum_{k=1}^{m} \mu^k A_k v. \end{cases} \tag{21}$$

This is a more general problem than studied in Section 3, but fortunately the μ-part of Theorem 3 can be generalized in the following sense. Motivated by (21), we study the following more general polynomial two-parameter eigenvalue problem:

$$\begin{cases} A_1 x = \lambda \sum_{k=0}^{m} \mu^k B_{1,k} x + \sum_{k=1}^{m} \mu^k C_{1,k} x, \\ A_2 y = \lambda \sum_{k=0}^{m} \mu^k B_{2,k} y + \sum_{k=1}^{m} \mu^k C_{2,k} y. \end{cases} \tag{22}$$

Associated with this polynomial two-parameter eigenvalue problem is the following polynomial eigenvalue problem (PEP) for μ.

Theorem 4 *If (λ, μ) is an eigenvalue of (22) with eigenvector (x, y) then*

$$\Big[(A_1 \otimes B_{2,0} - B_{1,0} \otimes A_2) + \sum_{k=1}^{m} \mu^k (A_1 \otimes B_{2,k} - B_{1,k} \otimes A_2 - C_{1,k} \otimes B_{2,0} + B_{1,0} \otimes C_{2,k}) + \sum_{k=1,i=1}^{m} \mu^{k+i} (B_{1,k} \otimes C_{2,i} - C_{1,k} \otimes B_{2,i})\Big] (x \otimes y) = 0.$$

Proof. One may check that

$$\begin{cases} A_1 x = \lambda B_1 x + \mu C_1 x + \lambda\mu D_1 x, \\ A_2 y = \lambda B_2 y + \mu C_2 y + \lambda\mu D_2 y \end{cases}$$

if we let $B_i = B_{i,0}$, $D_i = \sum_{k=1}^{m} \mu^{k-1} B_{i,k}$ and $C_i = \sum_{k=1}^{m} \mu^{k-1} C_{i,k}$ for $i = 1, 2$. Application of Theorem 3 yields that

$$\begin{aligned} 0 = \Big[&(A_1 \otimes B_{2,0} - B_{1,0} \otimes A_2) \\ &+\mu \Big(A_1 \otimes \sum_{k=1}^{m} \mu^{k-1} B_{2,k} - \sum_{k=1}^{m} \mu^{k-1} B_{1,k} \otimes A_2 \\ &\quad - \sum_{k=1}^{m} \mu^{k-1} C_{1,k} \otimes B_{2,0} + B_{1,0} \otimes \sum_{k=1}^{m} \mu^{k-1} C_{2,k} \Big) \\ &+\mu^2 \Big(\sum_{k=1}^{m} \mu^{k-1} B_{1,k} \otimes \sum_{k=1}^{m} \mu^{k-1} C_{2,k} - \sum_{k=1}^{m} \mu^{k-1} C_{1,k} \otimes \sum_{k=1}^{m} \mu^{k-1} B_{2,k} \Big) \Big] (x \otimes y) \\ = \Big[&(A_1 \otimes B_{2,0} - B_{1,0} \otimes A_2) \\ &+\Big(A_1 \otimes \sum_{k=1}^{m} \mu^{k} B_{2,k} - \sum_{k=1}^{m} \mu^{k} B_{1,k} \otimes A_2 \\ &\quad - \sum_{k=1}^{m} \mu^{k} C_{1,k} \otimes B_{2,0} + B_{1,0} \otimes \sum_{k=1}^{m} \mu^{k} C_{2,k} \Big) \\ &+\Big(\sum_{k=1}^{m} \mu^{k} B_{1,k} \otimes \sum_{k=1}^{m} \mu^{k} C_{2,k} - \sum_{k=1}^{m} \mu^{k} C_{1,k} \otimes \sum_{k=1}^{m} \mu^{k} B_{2,k} \Big) \Big] (x \otimes y), \end{aligned}$$

which completes the proof. □

We see that (21) is of the same form as the polynomial two-parameter eigenvalue problem in (22) with $A_1 = -\overline{A}_m$, $B_{1,m} = \overline{B}_0$, $B_{1,k} = 0$, $k = 0, \ldots, m-1$, $C_{1,k} = \overline{A}_{m-k}$, $k = 1, \ldots, m$, $A_2 = A_0$, $B_{2,0} = B_0$, $B_{2,k} = 0$, $k = 1, \ldots, m$, $C_{2,k} = -A_k$, $k = 1, \ldots, m$.

Theorem 4 and several manipulations of the sums yield

$$\begin{aligned} 0 = \Big[&- \overline{A}_m \otimes B_0 + \mu^m (-\overline{B}_0 \otimes A_0) + \sum_{k=1}^{m} \mu^k (-\overline{A}_{m-k} \otimes B_0) \\ &\qquad + \sum_{i=1}^{m} \mu^{m+i} (-\overline{B_0} \otimes A_i) \Big] (u \otimes v) \\ = \Big[&- \sum_{k=0}^{m} \mu^{m-k} (\overline{A}_k \otimes B_0) - \sum_{i=0}^{m} \mu^{m+i} (\overline{B}_0 \otimes A_i) \Big] (u \otimes v). \end{aligned}$$

This is a slightly more general equation than the polynomial eigenproblem in [15]. If the matrices are real and $B_0 = I$, then we have the polynomial eigenproblem [15, Eq. (14)].

As in Section 2 and the neutral case in the previous subsection, we may consider the conjugate transpose instead of the conjugate. More precisely, if we switch the two equations in (21), replace conjugation by the conjugate transpose and apply Theorem 4 the equation in μ is the polynomial eigenvalue problem in [4, Thm. 3.1] (again assuming that the matrices are real and $B_0 = I$). The resulting equation is left out for brevity, since a more general result is given next.

Finally, the most general result is for neutral commensurate DDEs. We show that the eigenvalue problem in Fu, Niculescu, and Chen [8] also is a polynomial eigenvalue problem that is connected with a polynomial two-parameter eigenvalue problem. Although the analysis is similar to that of the previous cases, this general case involves more technicalities and more involved expressions. Consider the polynomial two-parameter eigenvalue problem corresponding to the neutral commensurate DDE

$$\sum_{k=0}^{m} B_k \dot{x}(t-\tau k) = \sum_{k=0}^{m} A_k x(t-\tau k),$$

i.e.,

$$\left(A_0 - \sum_{k=0}^{m} \lambda \mu^k B_k + \sum_{k=1}^{m} \mu^k A_k \right) x = 0. \tag{23}$$

The complex conjugate transpose is

$$\left(\mu^m A_0^* + \sum_{k=0}^{m} \lambda \mu^{m-k} B_k^* + \sum_{k=1}^{m} \mu^{m-k} A_k^* \right) y = 0. \tag{24}$$

We can now combine (23) and (24) into a polynomial two-parameter eigenvalue problem

$$\begin{cases} A_0 x = \lambda \sum_{k=0}^{m} \mu^k B_k x - \sum_{k=1}^{m} \mu^k A_k x, \\ -A_m^* y = \lambda \sum_{k=0}^{m} \mu^k B_{m-k}^* y + \sum_{k=1}^{m} \mu^k A_{m-k}^* y. \end{cases} \tag{25}$$

This corresponds to (22) with $A_1 = A_0$, $B_{1,k} = B_k$, $k = 0, \dots, m$, $C_{1,k} = -A_k$, $k = 1, \dots, m$, $A_2 = -A_m^*$, $B_{2,k} = B_{m-k}^*$, $k = 1, \dots, m$, $C_{2,k} = A_{m-k}^*$, $k = 1, \dots, m$. Theorem 4 yields

$$
\begin{aligned}
&\Big[(A_0 \otimes B^*_{m-0} + B_0 \otimes A^*_m) \\
&\quad + \sum_{k=1}^{m} \mu^k (A_0 \otimes B^*_{m-k} + B_k \otimes A^*_m + A_k \otimes B^*_{m-0} + B_0 \otimes A^*_{m-k}) \\
&\quad + \sum_{k=1,i=1}^{m} \mu^{k+i} (B_k \otimes A^*_{m-i} + A_k \otimes B^*_{m-i})\Big] (x \otimes y) = 0.
\end{aligned}
$$

We note that with some effort it can be verified that the matrix coefficients Q_k in [8, Thm. 2] are exactly the matrix coefficients that occur in this polynomial eigenvalue problem.

5 New variants of matrix pencil methods

In this section, we introduce some new matrix pencil methods, which are variants of existing approaches. For ease of presentation, we will state the results for neutral DDEs with one delay, but all methods can be generalized for DDEs with multiple commensurate delays.

Moreover, we will only mention the relevant quadratic two-parameter eigenvalue problems (which will be polynomial two-parameter eigenvalue problems for DDEs with multiple commensurate delays); as we have seen before, every such two-parameter eigenvalue problem has two associated (one-parameter) eigenvalue problems, one for λ and one for μ giving two possible resulting matrix pencil methods. (For DDEs with multiple commensurate delays there seems to be just one natural associated polynomial eigenproblem, for μ.)

The quadratic two-parameter eigenvalue problem for the neutral single-delay DDE given in (18) is just one of several possible quadratic two-parameter eigenvalue problems. We can get the following expressions by transposing none, one, or both equations:

a) (18);

b) (18) but with the first equation transposed:

$$
\begin{cases}
A_0^T x = \lambda B_0^T x + \lambda\mu B_1^T x - \mu A_1^T x, \\
\overline{A}_1 y = -\lambda \overline{B}_1 y - \lambda\mu \overline{B}_0 y - \mu \overline{A}_0 y;
\end{cases}
$$

c) (18) but with the second equation transposed:

$$
\begin{cases}
A_0 x = \lambda B_0 x + \lambda\mu B_1 x - \mu A_1 x, \\
A_1^* y = -\lambda B_1^* y - \lambda\mu B_0^* y - \mu A_0^* y;
\end{cases}
\tag{26}
$$

d) and (18) but with both equations transposed:

$$\begin{cases} A_0^T x = \quad \lambda B_0^T x + \lambda\mu B_1^T x - \mu A_1^T x, \\ A_1^* y = -\lambda B_1^* y - \lambda\mu B_0^* y - \mu A_0^* y. \end{cases}$$

Applying any of the two parts of Theorem 3 yields an associated GEP corresponding to a matrix pencil method. As an additional permutation, the order of the two equations in a two-parameter eigenvalue problem does not influence the problem and can be interchanged to yield yet other variants. Hence, in total for the neutral single delay DDE we find $4 \cdot 2 \cdot 2 = 16$ matrix pencil variants. (For DDEs with multiple commensurate delays there appears to be just one natural associated polynomial eigenproblem, for μ, resulting in 8 variants.)

The methods known in the literature correspond to the following:

- [4] and [8]: (26) and the μ-part of Theorem 3;
- [19]: (18) and the λ-part of Theorem 3;
- [15], [13], and [6]: (18) and the μ-part of Theorem 3.

Finally, we stress that the above list, which contains many new variants, is more than just an theoretical encyclopedic description of all possible options. Depending on the given matrices, the structure and sparsity patterns of the Kronecker products may differ which may imply that for certain applications some methods may be more favorable than others.

6 Conclusions and outlook

We have recognized new types of eigenvalue problems: quadratic and polynomial two-parameter eigenvalue problems. Using these problems as a unifying framework, we have derived associated (one-parameter) quadratic or polynomial eigenvalue problems that are at the heart of many matrix pencil methods that are used to analyze asymptotic stability of DDEs. This unifying way to derive the matrix pencils in the matrix pencil methods provides further understanding of these methods and makes it easier to compare various approaches. Moreover, we have proposed several new variations on known matrix pencil methods.

Furthermore, we expect that the recognized framework of quadratic and polynomial two-parameter eigenvalue problem may lead to a considerable amount of new research. First, we want to stress that it has been outside of the scope of this paper to study theoretical and practical properties of these new types of eigenvalue problems. There are many interesting aspects that need further investigation, such as how to carry over the concept of linearization (as is common practice for QEPs and PEPs) to these problems.

Second, the matrix pencils constructed by Kronecker products that occur in the matrix pencil methods are of large dimension by nature, even for medium-sized problems, which may make efficient computation of eigenvalues and stability of DDEs very challenging. We believe that the key to a successful computational approach lies in a direct attack of the polynomial two-parameter eigenvalue problem, instead of the corresponding matrix pencils, in the same spirit as, for instance, [12] for the linear two-parameter eigenproblem. We leave both of these topics for future work. Third, we note that a recent preprint by Muhič and Plestenjak examines interesting relations between quadratic two-parameter eigenvalue problems and singular linear two-parameter eigenvalue problems [23].

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